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Electro-magnetic waves within a model for charged solitons

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Abstract
We analyse the model of topological fermions (MTF), where charged fermions are treated as soliton solutions of the field equations. In the region far from the sources we find plane waves solutions with the properties of electro-magnetic waves.

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(Some figures in this article are in colour only in the electronic version)

The intrinsic beauty of the Skyrme model and the well-known success of its application to short-range forces and strongly coupled particles make it worthwhile to extend its philosophy to the description of long-range forces and electrically coupled particles. The model of topological fermions (MTF) attempts to realize such an idea [1, 2].

The MTF field, $Q(x)$, is an $SU(2)$ field parameterized by

$$Q(x) = \cos\alpha(x) + \mathrm{i}\vec{\sigma}\vec{n}(x)\sin\alpha(x), \tag{1}$$

where $\vec{\sigma}$ are the Pauli matrices. The fields $\alpha(x)$ and $\vec{n}(x)$ are functions of the Minkowski coordinates $x^\mu = (ct, x, y, z)$. The $\vec{n}(x)$ field is a three-dimensional vector in internal ('colour') space[3]. It is constrained by condition $\vec{n}^2(x) = 1$ and defines a two-dimensional sphere which further on we call $\mathrm{S}^2_{\mathrm{col}}$.

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[3] We use the summation convention that any capital Latin index that is repeated in a product is automatically summed on from 1 to 3. The arrows on variables in the internal 'colour' space indicate the set of three elements $\vec{q} = (q_1, q_2, q_3)$ or $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ and $\vec{q}\vec{\sigma} = q_K\sigma_K$. We use the wedge symbol $\wedge$ for the external product between colour vectors $(\vec{q} \wedge \vec{\sigma})_A = \epsilon_{ABC}q_B\sigma_C$. For the components of vectors in physical space $\mathbf{x} = (x, y, z)$ we employ small Latin indices, i, j, k and a summation convention over doubled indices, e.g. $(\mathbf{E} \times \mathbf{B})_i = \epsilon_{ijk}E_jB_k$. Further we use the metric $\eta = \mathrm{diag}(1, -1, -1, -1)$ in Minkowski space.

The Lagrangian of the MTF reads

$$\mathcal{L} = -\frac{\alpha_f \hbar c}{4\pi}\left(\frac{1}{4}\vec{R}_{\mu\nu}\cdot\vec{R}^{\mu\nu} + \Lambda(q_0)\right), \tag{2}$$

where $\vec{R}^{\mu\nu}$ is curvature tensor

$$\vec{R}^{\mu\nu} = \vec{\Gamma}^{\mu} \wedge \vec{\Gamma}^{\nu}, \tag{3}$$

with the connection

$$\vec{\Gamma}^{\mu} = \frac{1}{2\mathrm{i}}\,\mathrm{Tr}(\vec{\sigma}\partial^{\mu} Q Q^{\dagger}) \tag{4}$$

and the potential term is given by

$$\Lambda(q_0) = \frac{1}{r_0^4}\left(\frac{\mathrm{Tr}\,Q}{2}\right)^{2m} = \frac{1}{r_0^4}\cos^{2m}\alpha(x), \qquad m = 1, 2, 3, \ldots \tag{5}$$

The model contains two parameters, the fine-structure constant, $\alpha_f = e_0^2/4\pi\varepsilon_0\hbar c \approx 1/137$, and a dimensional parameter r_0.

Note that 'the curvature term' $-\frac{1}{4}\vec{R}_{\mu\nu}\cdot\vec{R}^{\mu\nu}$ is proportional to the Skyrme term, but the so-called kinetic term does not enter the Lagrangian (2) in order to allow for electromagnetic fields and forces [2].

Due to its Lagrangian the MTF has different properties from the Skyrme model at $r \to \infty$ [1, 2]. In the Skyrme model the chiral field U approaches the trivial configuration, $U \to 1$. In the MTF the field configuration is determined by the potential minimum, i.e. $\alpha(x) = \frac{\pi}{2}$ at $r \to \infty$. As a result the Q field becomes nontrivial,

$$Q(x) = \mathrm{i}\vec{\sigma}\vec{n}(x) \qquad \text{at} \qquad r \to \infty. \tag{6}$$

The field $\alpha(x)$ describes the profile of a charged soliton with properties of an electron, whereas the field $\vec{n}(x)$ is related to the dual electromagnetic field strength [1, 2] by

$${}^{*}f_{\mu\nu}(x) = -\frac{e_0}{4\pi\varepsilon_0 c}[\partial_\mu\vec{n}(x)\wedge\partial_\nu\vec{n}(x)]\cdot\vec{n}(x). \tag{7}$$

The field strength $f_{\mu\nu}$ reads $f_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}{}^{*}f^{\rho\sigma}$ with $\epsilon^{0123} = 1$.

In the wave zone, where $\alpha(x) \to \pi/2$, the field $\vec{n}(x)$ should describe the free electromagnetic field. This can be shown by solving the equations of motion in the wave zone and comparing them with the solutions of Maxwell's equations.

In the wave zone the equations of motion for the field $\vec{n}(x)$, derived in [2], are

$$\partial^{\mu}\vec{n}\partial^{\nu}\{[\partial_\mu\vec{n}(x)\wedge\partial_\nu\vec{n}(x)]\cdot\vec{n}(x)\} = 0. \tag{8}$$

Due to the identity $\vec{n}\cdot\partial_\mu\vec{n} = 0$ these are two independent equations only.

The aim of this paper is to solve equations (8) and to show that these solutions behave like electromagnetic waves.

In terms of the vector field $\vec{n}(x)$ electric $\mathbf{E}$ and magnetic $\mathbf{B}$ fields are defined by

$$\begin{aligned} E_i &= \tfrac{1}{2}\kappa\epsilon_{ijk}(\partial_j\vec{n}\wedge\partial_k\vec{n})\cdot\vec{n}, \\ c^2 B_i &= \kappa(\partial_t\vec{n}\wedge\partial_i\vec{n})\cdot\vec{n}, \end{aligned} \tag{9}$$

where $\kappa = -e_0/(4\pi\varepsilon_0)$ in the International System of Units (SI) [3].

In the wave zone the electric and magnetic fields, propagating along the z-direction should satisfy the constraints [4]

$$\begin{aligned} E_z &= \kappa\vec{n}(\partial_x\vec{n}\wedge\partial_y\vec{n})\cdot\vec{n} = 0, \\ c^2 B_z &= \kappa(\partial_t\vec{n}\wedge\partial_z\vec{n})\cdot\vec{n} = 0. \end{aligned} \tag{10}$$

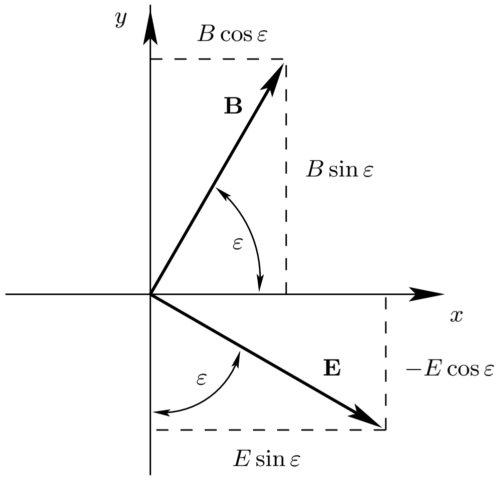

Figure 1. The physical meaning of the angle $\varepsilon(\zeta(z,t))$.

According to the constraint $\vec{n}^2 = 1$, the field $\vec{n}$ has two degrees of freedom. We describe these two degrees of freedom in terms of two variables, $\zeta(x^\mu)$ and $\eta(x^\mu)$. The constraints (10) can be fulfilled automatically by the following dependence of the $\vec{n}(x)$ field on ζ, η and the Minkowski coordinates:

$$\vec{n} = \vec{n}(\zeta(z,t), \eta(x, y, \zeta(z,t))). \tag{11}$$

Below we show that the variable $\zeta(z,t)$ is a function of $z \pm ct$ only.

In terms of ζ and η and using (11) the electric and magnetic fields (9) take the form

$$\begin{aligned} \mathbf{E} &= \kappa\vec{n}\cdot(\partial_\zeta\vec{n}\wedge\partial_\eta\vec{n})\partial_z\zeta(-\partial_y\eta, \partial_x\eta, 0), \\ c^2\mathbf{B} &= \kappa\vec{n}\cdot(\partial_\zeta\vec{n}\wedge\partial_\eta\vec{n})\partial_t\zeta(\partial_x\eta, \partial_y\eta, 0). \end{aligned} \tag{12}$$

Introducing the notations

$$\begin{aligned} &\cos\varepsilon = \frac{\partial_x\eta}{|\nabla_\perp\eta|}, \qquad \sin\varepsilon = \frac{\partial_y\eta}{|\nabla_\perp\eta|}, \\ &\text{where} \quad \nabla_\perp\eta \equiv (\partial_x\eta, \partial_y\eta), \end{aligned} \tag{13}$$

we get

$$\begin{aligned} \mathbf{E} &= \kappa\vec{n}\cdot(\partial_\zeta\vec{n}\wedge\partial_\eta\vec{n})\partial_z\zeta|\nabla_\perp\eta|(-\sin\varepsilon, \cos\varepsilon, 0), \\ c^2\mathbf{B} &= \kappa\vec{n}\cdot(\partial_\zeta\vec{n}\wedge\partial_\eta\vec{n})\partial_t\zeta|\nabla_\perp\eta|(\cos\varepsilon, \sin\varepsilon, 0). \end{aligned} \tag{14}$$

The parameter ε has the meaning of an angle between the x axis and the magnetic field (figure 1) [4].

Using the spherical angles θ and ϕ in colour space

$$n_x = \sin\theta\cos\phi, \qquad n_y = \sin\theta\sin\phi, \qquad n_z = \cos\theta \tag{15}$$

we can relate the factor $\vec{n}\cdot(\partial_\zeta\vec{n}\wedge\partial_\eta\vec{n})$ in equations (12) to the ratio of area elements between the two sets of internal coordinates $(\cos\theta, \phi)$ and (η, ζ). We will use an area preserving

[4] ε together with the propagation direction defines the polarization plane as was experimentally defined in crystal optics, 'Fresnel definition'.

mapping from (η, ζ) to $(\cos\theta, \phi)$ and get

$$\vec{n} \cdot (\partial_\zeta \vec{n} \wedge \partial_\eta \vec{n}) = \frac{\partial(\cos\theta, \phi)}{\partial(\eta, \zeta)} = 1. \tag{16}$$

For a special solution we will show later that this condition can result in topological restrictions on possible types of waves.

Of course such a mapping is not unique. This means that there may be different realizations for the non-observable $\vec{n}(x)$ field, which leads to the same physical fields **E** and **B**.

Assuming that the coordinates (η, ζ) fulfil condition (16) one arrives at

$$\mathbf{E} = \kappa \partial_z \zeta |\nabla_\perp \eta| (-\sin\varepsilon, \cos\varepsilon, 0), \qquad c^2\mathbf{B} = \kappa \partial_t \zeta |\nabla_\perp \eta| (\cos\varepsilon, \sin\varepsilon, 0). \tag{17}$$

The field strengths obviously fulfil the general condition for electromagnetic waves [4] $\mathbf{E}\cdot\mathbf{B} = 0$. In order to identify the electric and magnetic fields in equation (17) with those in electromagnetic waves they should satisfy another constraint, $|\mathbf{E}| = c|\mathbf{B}|$. According to equation (17) this leads to the relation $|\partial_z\zeta| = |\partial_t\zeta|/c$. This constraint is obviously fulfilled if ζ depends on $z_\pm = z \pm ct$. In order to show this we have to turn to the equations of motion (8). Substituting (11) into equations of motion (8) one arrives at two coupled nonlinear equations

$$\begin{aligned} &\partial_\zeta (\nabla_\perp \eta)^2 \left[(\partial_z \zeta)^2 - \frac{1}{c^2} (\partial_t \zeta)^2 \right] + 2(\nabla_\perp \eta)^2 \left[\partial_z^2 \zeta - \frac{1}{c^2} \partial_t^2 \zeta \right] = 0, \\ &\Delta_\perp \eta \left[(\partial_z \zeta)^2 - \frac{1}{c^2} (\partial_t \zeta)^2 \right] = 0, \\ &\Delta_\perp \eta \equiv \left(\partial_x^2 + \partial_y^2 \right) \eta. \end{aligned} \tag{18}$$

These equations can be fulfilled by the solutions of

$$\begin{cases} \partial_z^2 \zeta - \dfrac{1}{c^2} \partial_t^2 \zeta = 0, \\ (\partial_z \zeta)^2 - \dfrac{1}{c^2} (\partial_t \zeta)^2 = 0. \end{cases} \tag{19}$$

The first equation of the system (19) is a usual wave equation with two partial solutions

$$\zeta(z, t) = \zeta_+(z_+), \qquad \text{or} \qquad \zeta(z, t) = \zeta_-(z_-), \tag{20}$$

where ζ_+ and ζ_- are arbitrary functions. The second equation in (19) is a nonlinear one. The partial solutions ζ_+ and ζ_-, but not a superposition of them, satisfy this equation too. Thus we have shown that the constraint $|\partial_z\zeta| = |\partial_t\zeta|/c$ is fulfilled.

Hence, the electric and magnetic fields, given by equation (17), describe polarized 'electromagnetic waves' with amplitudes $\kappa\partial_z\zeta|\nabla_\perp\eta|$ depending on the space-time point. This dependence is a consequence of the nonlinearity of the equations of motion.

For the special case

$$\partial_\zeta (\nabla_\perp \eta)^2 = 0 \qquad \text{and} \qquad \Delta_\perp \eta = 0 \tag{21}$$

the system (18) is reduced to one equation, which is the wave equation $\partial_z^2\zeta - \frac{1}{c^2}\partial_t^2\zeta = 0$. The general solution for ζ is a superposition of two waves

$$\zeta(z, t) = \zeta_+(z_+) + \zeta_-(z_-). \tag{22}$$

Now let us discuss a special solution of equations (21)

$$\eta = a(\zeta)x + b(\zeta)y, \qquad a(\zeta) = d^{-1}\cos\varepsilon(\zeta), \qquad b(\zeta) = d^{-1}\sin\varepsilon(\zeta), \tag{23}$$

where d is some constant. It results in waves, which are independent of the x and y coordinates, like plane waves in Maxwell electrodynamics.

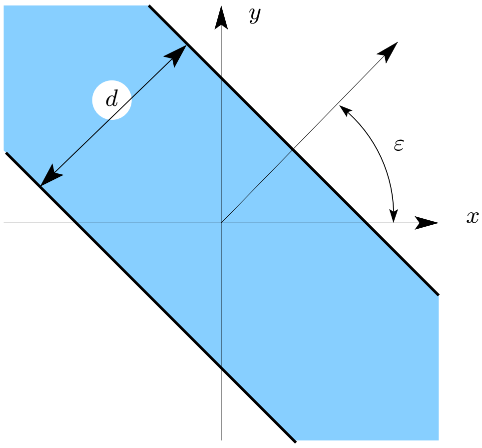

Figure 2. The shaded region indicates the area in the (x, y)-plane, where circular polarized waves exist.

Putting

$$\varepsilon = \text{const} \qquad \text{and} \qquad \zeta = \cos kz_\pm \tag{24}$$

one gets linear polarized electromagnetic waves

$$\mathbf{E} = \kappa\frac{k}{d}\sin kz_\pm(\sin\varepsilon, -\cos\varepsilon, 0), \qquad c\mathbf{B} = \mp\kappa\frac{k}{d}\sin kz_\pm(\cos\varepsilon, \sin\varepsilon, 0), \tag{25}$$

where k is the wave number. Now we have to show that a mapping to the coordinates $(\cos\theta, \phi)$ exists, which satisfies condition (16). Such a mapping is

$$\cos\theta = \zeta, \qquad \phi = -\eta. \tag{26}$$

For circular polarized waves the situation is not so simple. In this case the polarization angle ε is a linear function of $z_\pm$, $\varepsilon = z_\pm k$, and the absolute value of the electric and magnetic fields is constant. With the choice

$$\cos\theta = \eta = \frac{1}{d}(x\cos\varepsilon + y\sin\varepsilon), \qquad \phi = \zeta = z_\pm/l, \qquad \varepsilon = kz_\pm, \tag{27}$$

where d and l are arbitrary length parameters, we arrive at the required behaviour of $\mathbf{E}$ and $\mathbf{B}$

$$\mathbf{E} = \kappa/(dl)(-\sin kz_\pm, \cos kz_\pm, 0), \qquad c\mathbf{B} = \pm\kappa/(dl)(\cos kz_\pm, \sin kz_\pm, 0). \tag{28}$$

From (27) follows that the region of η is restricted to

$$-1 \leqslant \eta \leqslant +1. \tag{29}$$

This means that the wave exists only on a strip of width d in the (x, y)-plane parallel to the $\mathbf{E}$ field (filled region in figure 2). For given field strength E_0 the width d can be chosen arbitrarily large.

Let us investigate the topological necessity for the restriction (29) in more detail. We have a similar problem as mapping a globe onto a flat surface. In our case the globe corresponds to S^2_{col}, the surface is a two-dimensional area perpendicular to the electric flux lines (28). Such an area is the helicoidal area

$$x\sin kz_\pm - y\cos kz_\pm = 0 \quad \text{at} \quad t = 0; \tag{30}$$

it has the topology of R^2. The electric field strength is given by the ratio of an infinitesimal area on S^2_{col} to the corresponding area on the helicoid (see equation (9) and [1, 2]). Requiring constant field strength on the helicoid the mapping has to be area preserving. It is well known that an area preserving mapping of a globe onto a plain is only possible in a restricted region, e.g. that defined in equation (29), $-1 \leqslant \cos\theta = \eta \leqslant +1$.

For linear polarized waves the electric field strength is not constant on the $z, r_\perp$-plane, it oscillates with z and the mapping (26) can be defined in the whole space.

Now let us discuss the connection of the equations of motion (8) with electrodynamics. Introducing the abbreviation,

$$\begin{aligned} g^\mu &= \kappa\partial_\nu\{[\partial_\mu\vec{n}(x) \wedge \partial_\nu\vec{n}(x)] \cdot \vec{n}(x)\} \\ &= (c\rho_{\text{mag}}, \mathbf{g}) = (c\nabla \cdot \mathbf{B}, -\nabla \times \mathbf{E} - \partial_t\mathbf{B}) \end{aligned} \tag{31}$$

the equations of motion (8) reduce to

$$\partial_\mu\vec{n}g^\mu = 0. \tag{32}$$

The quantity g_μ is obviously conserved, $\partial_\mu g^\mu = 0$, and looks formally like a magnetic current [5]. But there is the essential difference to Dirac's magnetic current that g_μ has no external source; it is a result of the non-Abelian nature of the colour field $\vec{n}(x)$.

Evidently, the equations of motion (32) are fulfilled by solutions of the homogeneous Maxwell equations, $g^\mu = 0$. But the inverse does not hold true. In the wave zone the MTF equations of motion (32) substitute the homogeneous Maxwell equations.

According to equations (15) and (16) the magnetic current reads in terms of the parameters η and ζ

$$\begin{aligned} \rho_{\text{mag}} &= \frac{\kappa}{c^2}\partial_t\zeta\,\Delta_\perp\eta, \\ g_x &= \kappa\left\{\frac{\partial^2\eta}{\partial x\partial\zeta}\left[(\partial_z\zeta)^2 - \frac{1}{c^2}(\partial_t\zeta)^2\right] + \partial_x\eta\left(\partial_z^2\zeta - \frac{1}{c^2}\partial_t^2\zeta\right)\right\}, \\ g_y &= \kappa\left\{\frac{\partial^2\eta}{\partial y\partial\zeta}\left[(\partial_z\zeta)^2 - \frac{1}{c^2}(\partial_t\zeta)^2\right] + \partial_y\eta\left(\partial_z^2\zeta - \frac{1}{c^2}\partial_t^2\zeta\right)\right\}, \\ g_z &= -\kappa\partial_z\zeta\,\Delta_\perp\eta. \end{aligned} \tag{33}$$

If there is no superposition of two waves ζ_+ and ζ_- , the x and y components of the magnetic current vanish, $g_x = g_y = 0$. This condition is fulfilled for linear and circular polarized waves, (25) and (28), respectively. In turn, if condition (23) is fulfilled, ρ_{mag} and g_z are also vanishing. The solution for linear polarized waves agrees with this condition at any point of physical space. But for circular polarized waves it is not so at the boundary of the strip of figure 2 and ρ_{mag} and g_z appear at this region.

In summary, we discuss solutions of the model of topological fermions in the wave zone. The equations of motion are nonlinear field equations. We find plane wave solutions of these equations with the properties of electromagnetic waves. We describe our solutions for linear and circular polarized waves and give topological reasons, why it is not possible to define circular polarized waves with a modulus of the electric and magnetic field strength constant everywhere in space time.

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